

CS 331, Fall 2026

Mini-lecture 5 (10/14)

Probability

Ω = outcome space

(today: discrete, continuous possible but more complicated)

π = distribution

(aka, measure, density, PDF, ...)

Axiom: $\sum_{x \in \Omega} \pi(x) = 1$

"Some outcome will happen"

Ex 1: $\Omega = \{H, T\}$

outcome of coin flip

$\pi_1 = 1/2, 1/2$

fair

$\pi_2 = 9/10, 1/10$

unfair

$$\text{Ex 2: } \Omega = \{HH, HT, TH, TT\}$$

$$\pi_3 = \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$$

$$\pi_4 = \frac{9}{20}, \frac{1}{20}, \frac{9}{20}, \frac{1}{20}$$

$$\pi_5 = \frac{1}{2}, 0, \frac{2}{5}, \frac{1}{10}$$

Random variable: function of outcome X

"anything that's determined by the outcome"

$$\text{Ex: } Y = \# \text{ heads in } X \sim \pi_3$$

$$Y = 0, 1, 2$$

$$\text{w.p. } \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$$

Special r.v.: indicator of event $E \subseteq \text{subset of outcomes } \Omega$

$$\text{Denoted by } 1(E) = \begin{array}{ll} 0 & \text{doesn't occur} \\ 1 & \text{occurs} \end{array}$$

Conditional distribution

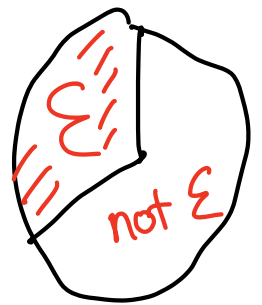
Let X have distribution π

Let $\mathcal{E}$ be some set of outcomes

Conditional dist: for all $x \in \mathcal{E}$

$$\pi(X=x | \mathcal{E}) = \frac{\pi(x)}{\sum_{x' \in \mathcal{E}} \pi(x')} \leftarrow \Pr[\mathcal{E}]$$

Interpretation: "re-expanding the pie"



Ex: X = outcome of dice roll

$\mathcal{E}$ = even rolls

$$\Pr[\mathcal{E}] = 1/2$$

$X = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

$$\pi(x) = 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6$$

$$\pi(x | \mathcal{E}) = 0 \quad 1/3 \quad 0 \quad 1/3 \quad 0 \quad 1/3$$

Sanity check: $\pi(x|\xi)$ is a distribution

Proof: $\sum_{x \in \Omega} \pi(x|\xi)$

$$= \sum_{x \in \xi} \pi(x|\xi) + \sum_{x \notin \xi} \pi(x|\xi)$$

$$= \sum_{x \in \xi} \frac{\pi(x)}{\Pr[\xi]} = \frac{\Pr[\xi]}{\Pr[\xi]} = 1 \quad \square$$

Joint distribution

Outcome space is product of two spaces

$$\Omega = \Omega_1 \times \Omega_2$$

outcomes: (x_1, x_2) e.g. H/T, H/T

From earlier... $\Omega = \{HH, HT, TH, TT\}$

$$\pi_3 = \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$$

$$\pi_4 = \frac{9}{20}, \frac{1}{20}, \frac{9}{20}, \frac{1}{20}$$

$$\pi_5 = \frac{1}{2}, 0, \frac{2}{5}, \frac{1}{10}$$

Marginal distributions:

We call μ_1 a X_1 -marginal of joint dist.

π on (X_1, X_2) if μ_1 is the dist.

After drawing $(X_1, X_2) \sim \pi$, "forget" X_2

Ex. X_1 marginal according to...

$$\pi_3 \Rightarrow \begin{array}{cc} H & \frac{1}{2} \\ T & \frac{1}{2} \end{array}$$

$$\pi_4 \Rightarrow \begin{array}{cc} H & \frac{1}{2} \\ T & \frac{1}{2} \end{array}$$

$$\pi_5 \Rightarrow \begin{array}{cc} H & \frac{1}{2} \\ T & \frac{1}{2} \end{array}$$

X_2 marginal according to...

$$\pi_3 \Rightarrow H \quad 1/2 \quad T \quad 1/2$$

$$\pi_4 \Rightarrow H \quad 9/10 \quad T \quad 1/10$$

$$\pi_5 \Rightarrow H \quad 9/10 \quad T \quad 1/10$$

We say $(X_1, X_2) \sim \pi$ are independent
if their marginals are μ_1, μ_2 ,

$$\pi(X_1, X_2) = \mu_1(X_1) \cdot \mu_2(X_2)$$

Q: Which of π_3, π_4, π_5 have independence?

Exercise: If $\Sigma \subseteq \Omega_2$, X_1, X_2 independent

$$\text{then } \mu_1(X_1 | \Sigma) = \mu_1(X_1)$$

(whatever happens to X_2 , doesn't affect X_1)

Puzzles

Q1: let $(X_1, X_2) \sim \pi$

w/ marginals μ_1, μ_2 that are fair dice

$$(\mu_1(i) = \mu_2(i) = 1/6 \text{ for } i = \begin{smallmatrix} 1, 2, \\ 3, 4, \\ 5, 6 \end{smallmatrix})$$

How large/small can $\Pr[X_1 = X_2]$ be?

Q2: let X_1, X_2 be indep. coins

(H/T w.p. $1/2$ each), let $\mathcal{E}$ = event

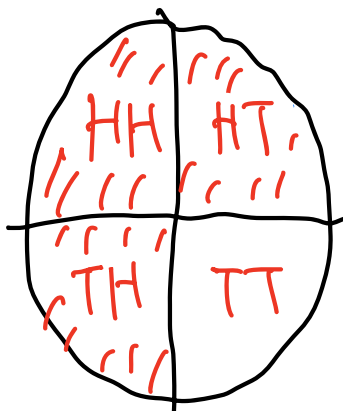
that at least one H. What is

$$\Pr[(X_1, X_2) = (H, H) \mid \mathcal{E}]?$$

A2: Math way:

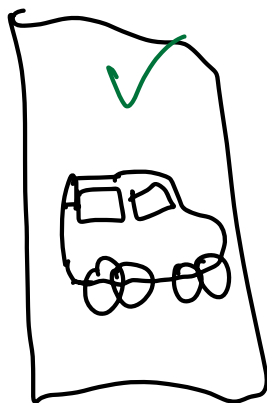
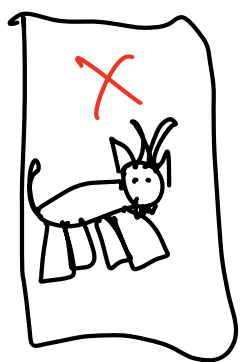
$$\Pr[(x_1, x_2) = (H, H) \mid \Sigma] \\ = \frac{\Pr(H, H)}{\Pr(\Sigma)} = \frac{1/4}{3/4} = 1/3.$$

Visual way:



Note: x_1, x_2
not indep. given Σ !

Q3 (Monty Hall):



Three doors

2 w/ goat

1 w/ car

(Randomly Chosen)

You choose a door. Then host shows you a goat.
Should you keep your door or switch?

A3: Say you chose door 1. (2,3 symmetric)

let $x \in \{1, 2, 3\}$ be where car is.

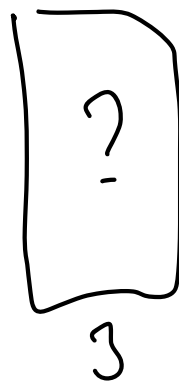
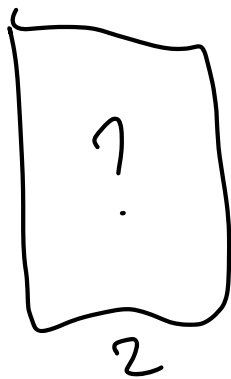
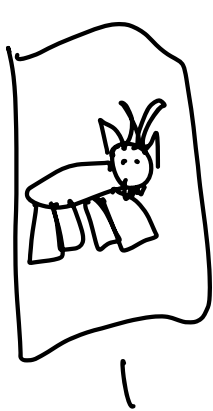
Ω , outcome space

let $W \subseteq \Omega$ be event "win if switch"

Observation: $W = \{2, 3\}$

Proof: if $x = 1$, any switch loses car.

if $x \neq 1$, world is like



host must reveal the other goat.

Hence, $\Pr[W] = \frac{2}{3}$, you should switch.

Expectation

Let $X \sim \pi$ have outcomes $\Omega \subset \mathbb{R}$

We define its expectation (average):

$$E[X] = \sum_{x \in \Omega} \pi(x) x$$

Ex. Let X be fair dice roll (6-sided)

$$E[X] = \frac{1}{6} \cdot (1+2+3+4+5+6) = 3.5$$

Ex. Let X be fair dice conditioned on even

$$E[X | \xi] = \frac{1}{3} \cdot (2+4+6) = 4$$

π_X is even

Other related concepts...

$$\text{Variance} = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Claim: $\text{Variance} \geq 0$ (we'll see why...)

Intuition: first term has no cancellations
second term does

$$\begin{aligned} \text{(e.g. } X = \begin{matrix} 1 \\ -1 \end{matrix} \text{ w.p. } 1/2, \quad \mathbb{E}[X^2] = 1 \\ \mathbb{E}[X]^2 = 0 \end{aligned} \rangle$$

It's used as a measure of "spread"

Why? We'll see in randomized unit...